

# MATH 303 – Measures and Integration

## Quiz Solutions and Explanations

Note: Some of the questions may be in a slightly different order than they appeared when you took the quiz. This is because some portions of the quiz were set to display the questions in a random order.

### 1 Domains of Measures, Premeasures, and Outer Measures

**Problem 1.** Match the type of object with the domain on which it is defined.

- Premeasure - **algebra**
- Measure -  **$\sigma$ -algebra**
- Outer Measure - **power set**

### 2 Properties of Measures, Premeasures, and Outer Measures

**Problem 2.** Select **all** of the properties that must be satisfied by a **measure**.

- ☒ assigns a value of 0 to the empty set
- ☒ nonnegative
- ☒ monotone
- ☒ countably additive
- ☒ countably subadditive

Explanation: By definition, a measure is a **nonnegative** function that **assigns a value of 0 to the empty set** and is **countably additive**. The other two properties are a consequence of these three (see Proposition 2.15 in the lecture notes).

**Problem 3.** Select **all** of the properties that must be satisfied by a **premeasure**.

- ☒ assigns a value of 0 to the empty set
- ☒ nonnegative
- ☒ monotone
- ☒ countably additive
- ☒ countably subadditive

Explanation: By definition, a premeasure is a **nonnegative** function that **assigns a value of 0 to the empty set** and is **countably additive**. Once again, the other two properties are a consequence of these three.

Another point of view on this problem is that a premeasure always extends to a measure by the Hahn–Kolmogorov extension theorem (Theorem 5.26 in the lecture notes), so it will have all of the same properties as a measure except for having a more limited domain.

**Problem 4.** Select **all** of the properties that must be satisfied by an **outer measure**.

- ☒ assigns a value of 0 to the empty set
- ☒ nonnegative
- ☒ monotone
- ☐ countably additive
- ☒ countably subadditive

Explanation: By definition, an outer measure is a **nonnegative** function that **assigns a value of 0 to the empty set** and is **monotone** and **countably subadditive**. It is possible for an outer measure to also be countably additive, but this is usually not the case. In order to obtain countable additivity, the outer measure should be restricted to the  $\sigma$ -algebra of measurable sets.

### 3 Measurable Functions

**Problem 5.** Let  $(X, \mathcal{B})$ ,  $(Y, \mathcal{C})$ , and  $(Z, \mathcal{D})$  be measurable spaces, and let  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$ . **True or False:** If  $f$  and  $g$  are measurable functions, then the composition  $g \circ f$  is also a measurable function.

- ☒ True
- ☐ False

Explanation: See Proposition 2.8.

**Problem 6.** Let  $(X, \mathcal{B})$ ,  $(Y, \mathcal{C})$ , and  $(Z, \mathcal{D})$  be measurable spaces, and let  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$ . **True or False:** If  $g \circ f$  is a measurable function, then  $f$  and  $g$  are measurable functions.

- ☐ True
- ☒ False

Explanation: If  $f$  is a constant function, then the composition  $g \circ f$  will also be constant and therefore measurable, regardless of measurability of  $g$ . The same is true if  $g$  is a constant function.

It is also possible to construct an example where neither  $f$  nor  $g$  is measurable. For instance, viewing  $\mathbb{R}$  as a vector space over  $\mathbb{Q}$ , a change of basis can be a nonmeasurable function (say, with respect to the Borel  $\sigma$ -algebra), and one can take  $f$  to be such a change of basis and  $g$  to be the inverse of  $f$ .

A simpler and more concrete example is to take  $X = Y = Z = \{a, b, c\}$  with the  $\sigma$ -algebra  $\mathcal{B} = \mathcal{C} = \mathcal{D} = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}\}$ . Let  $f$  be the permutation  $f(a) = b$ ,  $f(b) = c$ , and  $f(c) = a$ . Then  $f$  is nonmeasurable, since  $f^{-1}(\{a\}) = \{c\} \notin \mathcal{B}$  and  $f^{-1}(\{b, c\}) = \{a, b\} \notin \mathcal{B}$ . Let  $g$  be the inverse of  $f$ ; that is,  $g(a) = c$ ,  $g(b) = a$ , and  $g(c) = b$ . Then  $g$  is also nonmeasurable, since  $g^{-1}(\{a\}) = \{b\} \notin \mathcal{C}$  and  $g^{-1}(\{b, c\}) = \{a, c\} \notin \mathcal{C}$ . However, the composition  $g \circ f$  is the identity map, which is measurable.

## 4 Lebesgue Measurable Functions

**Problem 7.** We say that a real-valued function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is **Lebesgue-measurable** if it is measurable as a map from the measurable space  $(\mathbb{R}, \mathcal{M})$  to the measurable space  $(\mathbb{R}, \text{Borel}(\mathbb{R}))$ , where  $\mathcal{M}$  is the  $\sigma$ -algebra of Lebesgue-measurable sets. This means (**fill in the blanks** by clicking and dragging the appropriate option at the bottom):

For every Borel set  $E \subseteq \mathbb{R}$ , the set  $f^{-1}(E)$  is Lebesgue-measurable.

Explanation: The definition of a measurable function between two measurable spaces is Definition 2.7 in the lecture notes.

**Problem 8.** We say that a real-valued function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is **Lebesgue-measurable** if it is measurable as a map from the measurable space  $(\mathbb{R}, \mathcal{M})$  to the measurable space  $(\mathbb{R}, \text{Borel}(\mathbb{R}))$ , where  $\mathcal{M}$  is the  $\sigma$ -algebra of Lebesgue-measurable sets.

Let  $f, g : \mathbb{R} \rightarrow \mathbb{R}$ . **True or False:** If  $f$  and  $g$  are Lebesgue-measurable functions, then the composition  $g \circ f : \mathbb{R} \rightarrow \mathbb{R}$  is also Lebesgue-measurable.

☐ True

☒ False

Explanation: At first, this may seem to contradict Problem 5 above (which can be paraphrased as “the composition of measurable functions is measurable”), but there is a subtlety when working with Lebesgue-measurable functions. The issue in this problem is that we have put one  $\sigma$ -algebra (Lebesgue-measurable sets) on the domain of the function and a different  $\sigma$ -algebra (Borel sets) on the codomain. Giving a counterexample is still quite a challenge (mostly because it is difficult to construct functions that are not Lebesgue-measurable), but it can be done. An example is described in the first 4 pages of these lecture notes: <https://www.math.purdue.edu/~torresm/lecture-notes/lebesgue-theory/lebesgue-lecture17.pdf>

## 5 Integration

**Problem 9.** The monotone convergence theorem is a theorem about a sequence of functions  $(f_n)_{n \in \mathbb{N}}$  satisfying certain properties. Select **all** of the properties of  $(f_n)_{n \in \mathbb{N}}$  that are part of the hypothesis of the monotone convergence theorem.

☐  $f_n$  is continuous for each  $n$

☐  $f_n$  is differentiable for each  $n$

- ☐  $f_n$  is a simple function for each  $n$
- ☒  $f_n$  is a measurable function for each  $n$
- ☒  $f_n$  is nonnegative for each  $n$
- ☐  $f_n$  is an increasing function for each  $n$
- ☒  $(f_n)_{n \in \mathbb{N}}$  is an increasing sequence of functions
- ☐  $(f_n)_{n \in \mathbb{N}}$  is a decreasing sequence of functions

Explanation: See Theorem 3.10 in the lecture notes (or Problem 11 below) for the statement of the monotone convergence theorem.

**Problem 10.** Let  $(f_n)_{n \in \mathbb{N}}$  be an increasing sequence of extended real-valued measurable functions. **True or False:** The pointwise limit  $f(x) = \lim_{n \rightarrow \infty} f_n(x)$  exists for every  $x$ , and  $f$  is a measurable function.

- ☒ True
- ☐ False

Explanation: For each  $x$ , the sequence  $(f_n(x))_{n \in \mathbb{N}}$  converges (as an extended real number) to its supremum:  $\lim_{n \rightarrow \infty} f_n(x) = \sup_{n \in \mathbb{N}} f_n(x)$ . The supremum of a sequence of extended real-valued measurable functions is measurable (see Proposition 2.11).

**Problem 11.** Fill in the blanks to complete the statement of the monotone convergence theorem.

**Monotone Convergence Theorem:** Let  $(X, \mathcal{B}, \mu)$  be a measure space. Let  $(f_n)_{n \in \mathbb{N}}$  be a(n) increasing sequence of nonnegative measurable functions on  $X$ . Then

$$\int_X \lim_{n \rightarrow \infty} f_n \, d\mu \equiv \lim_{n \rightarrow \infty} \int_X f_n \, d\mu.$$

**Problem 12.** Fatou's lemma is a statement about a sequence of functions  $(f_n)_{n \in \mathbb{N}}$  satisfying certain properties. Select **all** of the properties of  $(f_n)_{n \in \mathbb{N}}$  that are part of the hypothesis of Fatou's lemma.

- ☐  $f_n$  is continuous for each  $n$
- ☐  $f_n$  is a simple function for each  $n$
- ☒  $f_n$  is a measurable function for each  $n$
- ☒  $f_n$  is nonnegative for each  $n$
- ☐  $(f_n)_{n \in \mathbb{N}}$  is an increasing sequence of functions
- ☐  $(f_n)_{n \in \mathbb{N}}$  is a decreasing sequence of functions

Explanation: See Theorem 3.13 in the lecture notes (or Problem 13 below) for the statement of Fatou's lemma.

**Problem 13.** Fill in the blank to complete the statement of Fatou's Lemma.

**Fatou's Lemma:** Let  $(X, \mathcal{B}, \mu)$  be a measure space. Let  $(f_n)_{n \in \mathbb{N}}$  be a sequence of nonnegative measurable functions on  $X$ . Then

$$\int_X \liminf_{n \rightarrow \infty} f_n \, d\mu \boxed{\leq} \liminf_{n \rightarrow \infty} \int_X f_n \, d\mu.$$

**Problem 14.** Let  $(X, \mathcal{B}, \mu)$  be a measure space, and let  $(f_n)_{n \in \mathbb{N}}$  be a sequence of complex-valued integrable functions on  $X$ . **True or False:** If the pointwise limit  $f(x) = \lim_{n \rightarrow \infty} f_n(x)$  exists (as a complex number) for every  $x \in X$ , then  $f$  is integrable and  $\int_X f \, d\mu = \lim_{n \rightarrow \infty} \int_X f_n \, d\mu$ .

☐ True

☒ False

Explanation: There are several different ways that this can fail.

Let  $X = \mathbb{N}$  with  $\mathcal{B} = \mathcal{P}(\mathbb{N})$  and  $\mu : \mathcal{P}(\mathbb{N}) \rightarrow [0, \infty]$  the counting measure. The sequence  $f_n = \mathbb{1}_{\{n\}}$  converges pointwise to 0, which is integrable, but  $\int_X 0 \, d\mu = 0$ , while  $\int_X f_n \, d\mu = \mu(\{n\}) = 1$  for each  $n \in \mathbb{N}$ .

It is also possible for the function  $f$  to not be integrable. For example, let  $X = \mathbb{R}$ , let  $\mathcal{B} = \mathcal{M}$  be the  $\sigma$ -algebra of Lebesgue-measurable sets, and let  $\mu = \lambda$  be the Lebesgue measure. The sequence  $f_n(x) = e^{2\pi i x} \mathbb{1}_{[-n, n]}(x)$  converges pointwise to  $f(x) = e^{2\pi i x}$ . Each of the functions  $f_n$  is integrable with integral 0. However,  $\int_{\mathbb{R}} |f| \, d\lambda = \int_{\mathbb{R}} 1 \, d\lambda = \lambda(\mathbb{R}) = \infty$ , so  $f$  is not an integrable function.

**Problem 15.** Fill in the blanks to complete the statement of the dominated convergence theorem.

**Dominated Convergence Theorem:** Let  $(X, \mathcal{B}, \mu)$  be a measure space. Let  $(f_n)_{n \in \mathbb{N}}$  be a sequence of integrable functions on  $X$ . Suppose that  $(f_n)_{n \in \mathbb{N}}$  converges almost everywhere to a measurable function  $f : X \rightarrow \mathbb{C}$ , and there exists a nonnegative integrable function  $g : X \rightarrow [0, \infty)$  such that  $|f_n| \leq g$  almost everywhere for each  $n$ . Then

$$\int_X f \, d\mu \boxed{=} \lim_{n \rightarrow \infty} \int_X f_n \, d\mu.$$

## 6 $\pi$ -Systems and $\lambda$ -Systems

**Problem 16.** Select all properties that must be satisfied by a  $\pi$ -system  $\mathcal{P}$  on a set  $X$ .

☐ contains the empty set

☐ contains  $X$

- ☒ closed under finite intersections
- ☐ closed under finite unions
- ☐ closed under complements
- ☐ closed under relative complements: if  $E, F \in \mathcal{P}$  and  $E \subseteq F$ , then  $F \setminus E \in \mathcal{P}$
- ☐ closed under countable unions
- ☐ closed under countable disjoint unions
- ☐ closed under countable increasing unions

Explanation: By definition, a  $\pi$ -system is **closed under finite intersections**. To see that the other properties may fail, we consider examples of  $\pi$ -systems.

Let  $\mathcal{P}_0 = \{(-\delta, \delta) \subseteq \mathbb{R} : \delta > 0, \delta \in \mathbb{Q}\}$  be the family of symmetric open intervals around 0 with rational radius. Then  $\mathcal{P}_0$  is a  $\pi$ -system on  $\mathbb{R}$ , since the intersection of two such intervals will be equal to the interval of smaller radius. Moreover,  $\mathcal{P}_0$  does not contain  $\emptyset$  and does not contain  $\mathbb{R}$ . Every element of  $\mathcal{P}_0$  must contain 0, so  $\mathcal{P}_0$  is not closed under complements nor relative complements. Finally,  $\mathcal{P}_0$  is not closed under countable (increasing) unions: if  $\delta_n$  is a sequence of rational numbers increasing to  $\sqrt{2}$ , then  $\bigcup_{n \in \mathbb{N}} (-\delta_n, \delta_n) = (-\sqrt{2}, \sqrt{2}) \notin \mathcal{P}_0$ . Hence, the only additional properties from the list that  $\mathcal{P}_0$  possesses is that it is closed under finite unions and (vacuously) closed under countable disjoint unions.

To eliminate the remaining two properties, we may consider the family  $\mathcal{P}_{int}$  of left-open, right-closed intervals in  $\mathbb{R}$ . The union  $(0, 1] \cup (2, 3]$  is not an interval, so it does not belong to  $\mathcal{P}_{int}$ . Hence,  $\mathcal{P}_{int}$  is not closed under finite unions. This same example shows that  $\mathcal{P}_{int}$  is not closed under countable disjoint unions by letting  $A_1 = (0, 1]$ ,  $A_2 = (2, 3]$ , and  $A_n = \emptyset$  for  $n \geq 3$ .

**Problem 17.** Select all properties that must be satisfied by a  $\lambda$ -system  $\mathcal{L}$  on a set  $X$ .

- ☒ contains the empty set
- ☒ contains  $X$
- ☐ closed under finite intersections
- ☐ closed under finite unions
- ☒ closed under complements
- ☒ closed under relative complements: if  $E, F \in \mathcal{L}$  and  $E \subseteq F$ , then  $F \setminus E \in \mathcal{L}$
- ☐ closed under countable unions
- ☒ closed under countable disjoint unions
- ☒ closed under countable increasing unions

Explanation: We defined a  $\lambda$ -system to be a family of sets that **contains the empty set** and is **closed under complements** and **closed under countable disjoint unions**. In Proposition 5.8, we showed that an equivalent definition of a  $\lambda$ -system is to use the other three properties from

the list: **contains  $X$**  and is **closed under relative complements** and **closed under countable increasing unions**.

Let us consider the three unselected properties from the list. Note that since a  $\lambda$ -system is closed under complements, being closed under finite intersections and being closed under finite unions are equivalent properties (since  $A \cap B = (A^c \cup B^c)^c$  and  $A \cup B = (A^c \cap B^c)^c$ ). Also, since  $\lambda$ -systems contain the empty set, being closed under countable unions implies being closed under finite unions. Therefore, any example of a  $\lambda$ -system that is not closed under finite intersections will in fact be a  $\lambda$ -system that satisfies none of the three unselected properties.

Consider the  $\lambda$ -system

$$\mathcal{L} = \{\emptyset, \{a, b, c, d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}\}$$

of subsets of  $\{a, b, c, d\}$  with an even number of elements. This  $\lambda$ -system is not closed under finite intersections, since, for example,  $\{a, b\} \cap \{a, c\} = \{a\} \notin \mathcal{L}$ .

**Problem 18.** For each example below, select the correct option that applies **in general**. Click and drag the correct answer from the options at the bottom to fill in each blank.

- The family of open subsets of a topological space is a  $\pi$ -system.  
Explanation: One of the axioms of a topology is that the family of open sets is closed under finite intersections. It need not be a  $\lambda$ -system. For example, the open subsets of  $\mathbb{R}$  are not closed under complementation.
- The family of open neighborhoods of a point in a topological space is a  $\pi$ -system.  
Explanation: Let  $x$  be a point in a topological space. If  $U$  and  $V$  are open neighborhoods of  $x$ , then (as in the first part)  $U \cap V$  is open and  $x \in U \cap V$ . The family of open neighborhoods of a point is never a  $\lambda$ -system, since it cannot contain the empty set.
- The family of left-open, right-closed intervals is a  $\pi$ -system.  
Explanation: It is a  $\pi$ -system since  $(a, b] \cap (c, d] = (\max\{a, c\}, \min\{b, d\}]$ . On the other hand,  $(0, 1]^c$  is not an interval, so the family of left-open, right-closed intervals is not a  $\lambda$ -system.
- A  $\sigma$ -algebra is both a  $\pi$ -system and a  $\lambda$ -system.  
Explanation: See Lemma 5.10.
- The family of sets  $\{E \in \mathcal{B} : \mu(E) = \nu(E)\}$  on which two probability measures  $\mu$  and  $\nu$  on a measurable space  $(X, \mathcal{B})$  agree is a  $\lambda$ -system.  
Explanation: Both measures give measure zero to the empty set, the measure of the complement of a set is  $1 - (\text{measure of the set})$ , and both  $\mu$  and  $\nu$  are countably additive, so the family of sets on which  $\mu$  and  $\nu$  agree satisfies all three axioms of a  $\lambda$ -system.  
 To see that this family does not have to be a  $\pi$ -system in general, consider the set  $X = \{a, b, c, d\}$  with the  $\sigma$ -algebra  $\mathcal{B} = \mathcal{P}(X)$ . Define probability measures  $\mu, \nu : \mathcal{P}(X) \rightarrow [0, 1]$  by  $\mu = \frac{1}{4}(\delta_a + \delta_b + \delta_c + \delta_d)$ , and  $\nu = \frac{1}{2}(\delta_a + \delta_c)$ . Let  $A = \{a, b\}$  and  $B = \{b, c\}$ . Then  $\mu(A) = \nu(A) = \frac{1}{2}$  and  $\mu(B) = \nu(B) = \frac{1}{2}$ , but  $\mu(A \cap B) = \frac{1}{4}$  while  $\nu(A \cap B) = 0$ .
- The family  $\{E \in \text{Borel}(\mathbb{R}) : \mu(E) = \nu(E)\}$  of Borel sets on which two Lebesgue–Stieltjes measures  $\mu$  and  $\nu$  agree is neither a  $\pi$ -system nor a  $\lambda$ -system.  
Explanation: Let us modify the example from the previous part. Let  $\mu = \delta_0 + \delta_1 + \delta_2$ , and let  $\nu = 2\delta_0 + 2\delta_2$ , and consider the family  $\mathcal{F} = \{E \in \text{Borel}(\mathbb{R}) : \mu(E) = \nu(E)\}$ .

Note that  $A = \{0, 1\}, B = \{1, 2\} \in \mathcal{F}$ , since  $\mu(A) = \nu(A) = 2$  and  $\mu(B) = \nu(B) = 2$ . However,  $A \cap B = \{1\} \notin \mathcal{F}$ , since  $\mu(\{1\}) = 1$  and  $\nu(\{1\}) = 0$ . Hence,  $\mathcal{F}$  is not a  $\pi$ -system.

Moreover,  $\mu(\mathbb{R}) = 3$  while  $\nu(\mathbb{R}) = 4$ , so  $\mathbb{R} \notin \mathcal{F}$ , which shows that  $\mathcal{F}$  is not a  $\lambda$ -system.

- The family  $\{E \in \mathcal{B} : \mu(E) = 0\}$  of null sets in a measure space  $(X, \mathcal{B}, \mu)$  is a  $\pi$ -system.  
Explanation: By monotonicity of measures, the intersection of two null sets is again a null set. If  $\mu$  is any measure with  $\mu(X) \neq 0$ , then the family of null sets is not a  $\lambda$ -system.

- The family  $\{E \in \mathcal{B} : \mu(X \setminus E) = 0\}$  of co-null sets in a measure space  $(X, \mathcal{B}, \mu)$  is a  $\pi$ -system.  
Explanation: Suppose  $E$  and  $F$  are co-null sets. Then  $X \setminus (E \cap F) = (X \setminus E) \cup (X \setminus F)$ , so by subadditivity,  $\mu(X \setminus (E \cap F)) \leq \mu(X \setminus E) + \mu(X \setminus F) = 0$ . That is,  $E \cap F$  is also co-null, so the family of co-null sets is a  $\pi$ -system.

Again, if  $\mu$  is any measure with  $\mu(X) \neq 0$ , then the family of co-null sets is not a  $\lambda$ -system (since  $\emptyset$  is not co-null).

- The family  $\{E \in \mathcal{B} : \mu(E) = 0 \text{ or } \mu(X \setminus E) = 0\}$  of null and co-null sets in a measure space  $(X, \mathcal{B}, \mu)$  is both a  $\pi$ -system and a  $\lambda$ -system.

Explanation: Checking that a family is both a  $\pi$ -system and a  $\lambda$ -system is equivalent to checking that it is a  $\sigma$ -algebra (see Lemma 5.10). Let  $\mathcal{C} = \{E \in \mathcal{B} : \mu(E) = 0 \text{ or } \mu(X \setminus E) = 0\}$ . We check each of the axioms of a  $\sigma$ -algebra:

- $\mu(X \setminus X) = \mu(\emptyset) = 0$ , so  $X \in \mathcal{C}$ .
- The condition for  $E$  to belong to  $\mathcal{C}$  is symmetric in  $E$  and  $E^c$ , so  $\mathcal{C}$  is closed under complements.
- Suppose  $(E_n)_{n \in \mathbb{N}}$  is a sequence of elements of  $\mathcal{C}$ , and let  $E = \bigcup_{n \in \mathbb{N}} E_n$ . If all of the sets  $E_n$  satisfy  $\mu(E_n) = 0$ , then  $\mu(E) \leq \sum_{n=1}^{\infty} \mu(E_n) = 0$ . On the other hand, if at least one of the sets  $E_{n_0}$  satisfies  $\mu(X \setminus E_{n_0}) = 0$  for some  $n_0 \in \mathbb{N}$ , then by monotonicity,  $\mu(X \setminus E) \leq \mu(X \setminus E_{n_0}) = 0$ . In either case, we conclude  $E \in \mathcal{C}$ .

- The family  $\{E \subseteq X : E \text{ or } X \setminus E \text{ is finite}\}$  of finite and co-finite subsets of an infinite set  $X$  is a  $\pi$ -system.

Explanation: Let  $\mathcal{F} = \{E \subseteq X : E \text{ or } X \setminus E \text{ is finite}\}$ , and let  $A, B \in \mathcal{F}$ . If either  $A$  or  $B$  is finite, then  $A \cap B$  is also finite, so  $A \cap B \in \mathcal{F}$ . If both  $A$  and  $B$  are co-finite, then  $X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$  is the union of two finite sets and therefore finite, so again  $A \cap B \in \mathcal{F}$ . This proves  $\mathcal{F}$  is a  $\pi$ -system.

To see that  $\mathcal{F}$  is not a  $\lambda$ -system, let  $(x_n)_{n \in \mathbb{N}}$  be a sequence of distinct elements of  $X$  (which exists since  $X$  is infinite). Then for each  $n \in \mathbb{N}$ , we have  $E_n = \{x_{2n}\} \in \mathcal{F}$ . However,  $E = \bigcup_{n \in \mathbb{N}} E_n = \{x_{2n} : n \in \mathbb{N}\}$  is an infinite set, and its complement contains the infinite set  $\{x_{2n-1} : n \in \mathbb{N}\}$ , so  $E \notin \mathcal{F}$ .

- The family  $\{E \subseteq X : E \text{ or } X \setminus E \text{ is countable}\}$  of countable and co-countable subsets of an uncountable set  $X$  is both a  $\pi$ -system and a  $\lambda$ -system.

Explanation: This was one of the first examples we gave of a  $\sigma$ -algebra (see Example 2.2).

**Problem 19.** Complete the statement of the  $\pi$ - $\lambda$  theorem.



$\pi$ - $\lambda$  **Theorem:** Let  $X$  be a set,  $\mathcal{P}$  a  $\pi$ -system on  $X$ , and  $\mathcal{L}$  a  $\lambda$ -system on  $X$ . Suppose  $\mathcal{P} \subseteq \mathcal{L}$ . Then  $\sigma(\mathcal{P}) \subseteq \mathcal{L}$ .

The  $\pi$ - $\lambda$  theorem is useful for proving uniqueness of measures satisfying certain properties.

Explanation: See Theorem 5.9. An example of its usefulness for establishing uniqueness of a measure is Corollary 5.14.

## 7 Radon Measures

**Problem 20.** Out of the following list of topological spaces, select **all** options that are **locally compact Hausdorff spaces**.

☒ Discrete spaces

Explanation: Every singleton is both open and compact, so points  $x \neq y$  can be separated by the open sets  $\{x\}$  and  $\{y\}$ , and every point  $x$  has compact neighborhood  $\{x\}$ .

☐ The rational numbers with the subspace topology inherited from the standard topology on  $\mathbb{R}$

Explanation: This is a Hausdorff space, but it is not locally compact. It can be checked the only compact subsets of  $\mathbb{Q}$  are finite sets, but every open set contains infinitely many points.

☒ Euclidean space  $\mathbb{R}^d$  for  $d \in \mathbb{N}$

Explanation: Two points  $x \neq y$  can be separated by taking an open ball around each point of radius smaller than  $|x - y|/2$ . Given  $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ , we have

$$x \in \underbrace{\prod_{j=1}^d (x_j - 1, x_j + 1)}_{U_x} \subseteq \underbrace{\prod_{j=1}^d [x_j - 1, x_j + 1]}_{K_x},$$

and  $U_x$  is open and  $K_x$  is compact (by the Heine–Borel theorem).

☒ Compact metric spaces

Explanation: Metric spaces are always Hausdorff: as in the previous example, given  $x \neq y$ , we can separate  $x$  and  $y$  by open balls of radius smaller than  $d(x, y)/2$ . If the whole space  $X$  is compact, then we can take  $X$  as the compact neighborhood of each point.

☐ The space  $C[0, 1]$  of complex-valued continuous functions  $f : [0, 1] \rightarrow \mathbb{C}$  with the topology of uniform convergence

Explanation: The topology is metrizable by taking the metric  $d(f, g) = \sup_{x \in [0, 1]} |f(x) - g(x)|$ , so  $C[0, 1]$  is a Hausdorff space. However, open sets are infinite-dimensional and therefore have non-compact closure.

**Problem 21.** Let  $X$  be an LCH space. A **Radon measure** is a Borel measure  $\mu : \text{Borel}(X) \rightarrow [0, \infty]$  satisfying 3 properties (click and drag the correct answer from the options at the bottom to fill in each blank):

1. locally finite: the measure of every compact set is finite

2. outer regular on  $\boxed{\text{Borel}}$  sets: if  $E$  is a(n)  $\boxed{\text{Borel}}$  set, then

$$\mu(E) = \boxed{\inf}\{\mu(F) : F \text{ is a(n) } \boxed{\text{open}} \text{ set and } E \boxed{\subseteq} F\}.$$

3. inner regular on  $\boxed{\text{open}}$  sets: if  $E$  is a(n)  $\boxed{\text{open}}$  set, then

$$\mu(E) = \boxed{\sup}\{\mu(F) : F \text{ is a(n) } \boxed{\text{compact}} \text{ set and } E \boxed{\supseteq} F\}.$$

Explanation: See Definition 6.4.

**Problem 22. True or False:** Every locally finite Borel measure on  $\mathbb{R}$  is a Radon measure.

☒ True

☐ False

Explanation: Locally finite measures on  $\mathbb{R}$  are Lebesgue–Stieltjes measures (restricted to the Borel  $\sigma$ -algebra), and we proved inner and outer regularity of Lebesgue–Stieltjes measures in Proposition 5.31.